3. $b : b'$ :: $4 : 3$. 3rd condition.

4. $v : v'$ :: $21 : 21$, multiplying and reducing, and remembering that the value $\propto l.b.q.$

5. Also $v - v' = \$500$. Whence,

6. $v = \$10500$. From (4) and (5),

7. $v' = \$10000$.

This problem was also solved by *B. F. SINE, NELSON S. RORAY, P. S. BERG, F. M. McGAW, J. C. CORBIN, COOPER D. SCHMITT, FREDERIC R. HONEY, H. C. WILKES,* and *G. B. M. ZERR.*
M. A. Gruber sent in a solution of Problem 70, Department of Arithmetic, too late for credit in last issue. His answer is 6.48 years.

ALGEBRA.

Conducted by **J. M. COLAW**, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

68. Proposed by **ROBERT JUDSON ALEY, M. A.**, Professor of Mathematics in Indiana University, Bloomington, Indiana.

Sum to n terms the series, $n\cos\theta + (n-1)\cos2\theta + (n-2)\cos3\theta$, etc.

[*Chrystal's Algebra.*]

I. Solution by **O. W. ANTHONY, M. Sc.**, Professor of Mathematics in Columbian University, Washington, D. C.

Let $S = n\cos\theta + (n-1)\cos2\theta + (n-2)\cos3\theta \ldots\ldots$ Also let $S_s = \sin\theta + \sin2\theta + \sin3\theta\ldots\ldots$, and $S_c = \cos\theta + \cos2\theta + \cos3\theta\ldots\ldots$

$$S = n[\cos\theta + \cos2\theta + \cos3\theta + \ldots\ldots] - [\cos2\theta + 2\cos3\theta + \ldots\ldots],$$

$$= (n+1)[\cos\theta + \cos2\theta + \cos3\theta + \ldots\ldots] - [\cos\theta + 2\cos2\theta + 3\cos3\theta \ldots\ldots],$$

$$= (n+1)S_c - dS_s/d\theta.$$

Now $S_c = [\cos\{\frac{1}{2}(n+1).\theta\}\sin\frac{1}{2}(n\theta)]/\sin\frac{1}{2}\theta$, and $S_s = [\sin\{\frac{1}{2}(n+1).\theta\}\sin\frac{1}{2}(n\theta)]/\sin\frac{1}{2}\theta$.

$$\therefore S = (n+1)\frac{\cos\{\frac{1}{2}(n+1)\theta\}\sin\frac{1}{2}(n\theta)}{\sin\frac{1}{2}\theta} - \frac{d}{d\theta}\left[\frac{\sin\frac{1}{2}(n+1).\theta\sin\frac{1}{2}(n\theta)}{\sin\frac{1}{2}\theta}\right],$$

probably as compact a form as can be obtained.

II. Solution by **G. B. M. ZERR, A. M., Ph. D.**, Texarkana, Arkansas.

Let $S = $ sum required,

$$2\sin\tfrac{1}{2}\theta\cos n\theta = \sin\{\theta + \frac{2n-1}{2}\theta\} - \sin\{\theta + \frac{2n-3}{2}\theta\}$$

$$4\sin\tfrac{1}{2}\theta\cos(n-1)\theta=2\sin\{\theta+\frac{2n-3}{2}\theta\}-2\sin\{\theta+\frac{2n-5}{2}\theta\}$$

$$6\sin\tfrac{1}{2}\theta\cos(n-2)\theta=3\sin\{\theta+\frac{2n-5}{2}\theta\}-3\sin\{\theta+\frac{2n-7}{2}\theta\}$$

$$8\sin\tfrac{1}{2}\theta\cos(n-3)\theta=4\sin\{\theta+\frac{2n-7}{2}\theta\}-4\sin\{\theta+\frac{2n-9}{2}\theta\}$$

. . . .

$$2n\sin\tfrac{1}{2}\theta\cos\theta=n\sin(\theta+\tfrac{1}{2}\theta)-n\sin(\theta-\tfrac{1}{2}\theta).$$

Adding we get

$$2S\sin\tfrac{1}{2}\theta=(\sin\tfrac{3}{2}\theta+\sin\tfrac{5}{2}\theta+\sin\tfrac{7}{2}\theta+\ldots\ldots+\sin\frac{2n+1}{2}\theta)-n\sin\tfrac{1}{2}\theta,$$

$$=[\sin(\frac{n+2}{2})\sin\tfrac{1}{2}(n\theta)]/\sin\tfrac{1}{2}\theta-n\sin\tfrac{1}{2}\theta.$$

$$\therefore\ S=[\sin(\frac{n+2}{2}\theta)\sin\tfrac{1}{2}(n\theta)-n\sin^2\tfrac{1}{2}\theta]/2\sin^2\tfrac{1}{2}\theta.$$

The series in parenthesis above is summed in all trigonometries in the series, $\sin\alpha+\sin(\alpha+\beta)+$, etc., by making $\alpha=\tfrac{3}{2}\theta$, $\beta=\theta$.

III. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The given series may be broken up into:

$$n[\cos\theta+\cos2\theta+\cos3\theta+\ldots\ldots\cos n\theta]$$
$$-[\cos2\theta+2\cos3\theta+3\cos4\theta+\ldots\ldots(n-1)\cos n\theta].$$

To sum the series $\cos\theta+\cos2\theta+\cos3\theta+\ldots\ldots\cos n\theta$, we have

$$\sin\tfrac{1}{2}\theta-\sin\tfrac{3}{2}\theta=-2\cos\theta\sin\tfrac{1}{2}\theta.$$
$$\sin\tfrac{3}{2}\theta-\sin\tfrac{5}{2}\theta=-2\cos2\theta\sin\tfrac{1}{2}\theta.$$
$$\sin\tfrac{5}{2}\theta-\sin\tfrac{7}{2}\theta=-2\cos3\theta\sin\tfrac{1}{2}\theta.$$

. . . .

$$\sin\tfrac{1}{2}(2n-1)\theta-\sin\tfrac{1}{2}(2n+1)\theta=-2\cos n\theta\sin\tfrac{1}{2}\theta.$$

Adding, we have, $\sin\tfrac{1}{2}\theta-\sin\tfrac{1}{2}(2n+1)\theta=-2\sin\tfrac{1}{2}\theta\Sigma(n\theta).$

$$\therefore\ \Sigma(n\theta)=[\sin\tfrac{1}{2}(2n+1)\theta-\sin\tfrac{1}{2}\theta]/2\sin\tfrac{1}{2}\theta=[\cos\tfrac{1}{2}(n+1)\theta\sin\tfrac{1}{2}n\theta]/\sin\tfrac{1}{2}\theta.$$

$$\therefore\ n\Sigma(n\theta)=[n\cos\tfrac{1}{2}(n+1)\theta\sin\tfrac{1}{2}n\theta]/\sin\tfrac{1}{2}\theta.$$

To sum the second part, we have,

$$x^2+2x^3+3x^4+\ldots\ldots(n-1)x^n=[x^2-nx^{n+1}+(n-1)x^{n+2}]/(1-x)^2.$$

Putting $x=\cos\theta+i\sin\theta$, and employing the formula $(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta$, we obtain after putting the real parts of both members equal, and making all necessary reductions, for the sum of the second series

$$=\frac{\cos\theta-n\cos n\theta+(n-1)\cos(n+1)\theta}{4\sin^2\frac{1}{2}\theta};$$

so that the sum of the given series

$$=\frac{n\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}+\frac{\cos\theta-n\cos n\theta+(n-1)\cos(n+1)\theta}{4\sin^2\frac{1}{2}\theta}.$$

To test this formula we must of course, leave the coefficient n of the first expression unchanged, while in all the other factors and terms which involve n, n must be put successively $=1, 2, 3, 4$, etc.

Also solved by *E. W. MORRELL.*

69. Proposed by C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that $x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}\pm\ldots\ldots\ldots+(\pm1)^nx$, where $A, B, C, \ldots\ldots$ are the binomial coefficients of the $(n+1)$th order.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

$$x^n\pm x^{n-1}+x^{n-2}+\ldots\ldots, \text{ etc.}=(x^{n+1}-1)/(x-1)\ldots\ldots\ldots\ldots\ldots(1),$$

or $x^{n+1}+1)/(x+1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2), =\{[(x-1)+1]^{n+1}-1\}/(x-1),$ or

$\{[(x+1)-1]^{n+1}+1\}/(x+1), =(x-1)^n+C^2_{n+1}(x-1)^{n-1}+C^3_{n+1}(x-1)^{n-2}+\ldots\ldots,$

or $(x+1)^n-C^2_{n+1}(x+1)^{n-1}+C^3_{n+1}(x-1)^{n-2}-\ldots\ldots,$

$$=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}+\ldots\ldots$$

II. Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

Let $K=x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n.$

Put $x=y\pm1$, expanding and observing that the sign of the last term of each expression is $\pm$ if n is odd but $+$ if n is even, we may write:

$$x^n=(y\pm1)^n=y^n\pm ny^{n-1}+\tfrac{1}{2}[n(n-2)]y^{n-2}\pm\ldots\ldots\ldots\ldots\ldots+(\pm1)^{n-1}ny+(\pm1)^n$$

$$\pm x^{n-1}=\pm(y\pm1)^{n-1}=\pm y^{n-1}+(n-1)y^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}(n-1)y+(\pm1)^n$$

$$x^{n-2}=(y\pm1)^{n-2}=\ldots\ldots\ldots\ldots\ldots y^{n-2}\pm\ldots\ldots\ldots\ldots+(\pm1)^{n-1}(n-2)y+(\pm1)^n$$

etc $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$etc.